

EINSTEIN MANIFOLDS OF DIMENSION FIVE WITH SMALL FIRST EIGENVALUE OF THE DIRAC OPERATOR

TH. FRIEDRICH & I. KATH

1. Introduction

Let M^n be a compact Einstein spin manifold with positive scalar curvature $R > 0$ and denote by $D: \Gamma(S) \rightarrow \Gamma(S)$ the Dirac operator acting on sections of the spinor bundle. If λ_1 is the first eigenvalue of this operator we have

$$\lambda_1^2 \geq \frac{1}{4} \frac{n \cdot R}{n-1}$$

(see e.g. [4]). Thus, there arises the interesting problem to classify all those Einstein spaces where the lower bound $\pm \frac{1}{2} \sqrt{\frac{(n \cdot R)}{(n-1)}}$ actually is an eigenvalue of the Dirac operator. The corresponding eigenspinor ψ satisfies the stronger equation

$$\nabla_X \psi = \mp \frac{1}{2} \sqrt{\frac{R}{n(n-1)}} X \cdot \psi$$

(see e.g. [4]) and these spinors are sometimes called Killing spinors (see e.g. [9], [16]). In case $n = 4$ the only possible manifold is $M^4 = S^4$ (see e.g. [5]).

In dimension six each solution of the equation $D\psi = \frac{1}{2} \sqrt{(6 \cdot R)/5} \psi$ defines a (nonintegrable) almost complex structure (see e.g. [8]). Furthermore, the assumption that $\pm \frac{1}{2} \sqrt{(n \cdot R)/(n-1)}$ is an eigenvalue of the Dirac operator imposes algebraic conditions on the Weyl tensor of the space (see e.g. [5]) as well as on the covariant derivative of the curvature tensor and the harmonic forms on M^n (see e.g. [9]). On the other hand, in the dimensions 5,6,7 examples of Einstein spaces different from the sphere are known for which $\pm \frac{1}{2} \sqrt{(n \cdot R)/(n-1)}$ is an eigenvalue of the Dirac operator (see e.g. [4], [7], [17]). Moreover, if M^n is a Kähler manifold, K.D. Kirchberg proved the stronger inequality $\lambda_1^2 \geq \frac{1}{4}(n+2)R/n$ (see e.g. [12]) and solved in the complex dimension $n/2 = 3$ the corresponding classification problem (see e.g. [13]); the only possible Einstein-Kähler spaces of complex dimension three realizing $\sqrt{\frac{2}{3}}R$ as an eigenvalue of the Dirac operator are $P^3(\mathbb{C})$ and $F(1,2)$

with their canonical metrics. The aim of this paper is to study the above mentioned classification problem in the case of 5-dimensional real Einstein spaces. First of all we prove that any solution of the equation $D\psi = \pm \frac{1}{4}\sqrt{5R}\psi$ defines an Einstein-Sasaki structure on M^5 . Conversely, if M^5 is a simply-connected Einstein-Sasaki space then the equation under consideration has a nontrivial solution. In the next step we classify all regular contact metric structures arising from a nontrivial solution of the equation $D\psi = \frac{1}{4}\sqrt{5R}\psi$. The regularity assumption implies that M^5 is a fiber bundle over a four-dimensional Einstein-Kähler manifold X^4 with positive scalar curvature. Therefore, we know the possible X^4 ($= S^2 \times S^2, P^2(\mathbb{C})$ or the del Pezzo surfaces $P_k, 3 \leq k \leq 8$) as well as the topological type of the fibration $\pi: M^5 \rightarrow X^4$. In particular, if M^5 is a simply-connected, compact 5-dimensional Einstein spin manifold such that $D\psi - \frac{1}{4}\sqrt{5R}\psi$ admits a nontrivial solution and the corresponding Sasaki structure is regular, then M^5 is isometric to the sphere S^5 , or to the Stiefel manifold $V_{4,2}$ with the Einstein metric considered in [11], [4], or M^5 is the simply-connected S^1 -bundle with Chern class $c_1^* = c_1(P_k)$ over one of the del Pezzo surfaces P_k ($3 \leq k \leq 8$). In the last case M^5 is diffeomorphic to the connected sum $M^5 \approx (S^2 \times S^3) \# \dots \# (S^2 \times S^3)$ and there is a one-to-one correspondence between Killing spinors on M^5 and Einstein-Kähler metrics on the del Pezzo surface P_k . The existence of Einstein-Kähler structures on P_k has been recently proved by Tian and Yau (see [21], [22]).

2. Einstein-Sasaki manifolds in dimension 5

We introduce some notation concerning contact structures. A general reference is [3]. A contact metric structure on a manifold M^5 consists of a 1-form η , a vector field ξ , a (1,1)-tensor φ and a Riemannian metric g such that the following conditions are satisfied:

- (a) $\eta \wedge (d\eta)^2 \neq 0$.
- (b) $\eta(\xi) = 1, \varphi(\xi) = 0$.
- (c) $\varphi^2 = -Id + \eta \otimes \xi$.
- (d) $g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y)$.
- (e) $d\eta(X, Y) = 2g(X, \varphi(Y))$ with $d\eta(X, Y) = X(\eta(Y)) - Y(\eta(X)) - \eta[X, Y]$.

Formal consequences of conditions (b) and (d) are the equations $\eta(X) = g(X, \xi), \varphi(\xi) = 0$.

In case ξ is a Killing vector field we call the given structure on M^5 a K -contact structure. This is equivalent to

- (f) $\nabla_X \xi = -\varphi(X)$.

A Sasaki manifold is a K -contact structure satisfying the integrability condition

$$[\varphi, \varphi] + d\eta\xi = 0$$

or, equivalently,

$$(g) (\nabla_X \varphi)(Y) = g(X, Y)\xi - \eta(Y)X.$$

The curvature tensor of a Sasaki manifold commutes with φ and has the following special property:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

In particular, if M^5 is a 5-dimensional Einstein-Sasaki manifold we obtain for the scalar curvature the value $R = 20$, and the Weyl tensor W satisfies $W(X, Y)\xi = 0$. Denote by $T^h \subset T(M^5)$ the bundle of all vectors orthogonal to ξ . According to $W(X, Y)\xi = 0$ we can consider the Weyl tensor of M^5 as a linear transformation

$$W: \bigwedge^2(T^h) \rightarrow \bigwedge^2(T^h).$$

T^h is an oriented 4-dimensional bundle and, consequently, we have the algebraic Hodge operator $*$: $\bigwedge^2(T^h) \rightarrow \bigwedge^2(T^h)$, obviously different from the Hodge operator of M^5 .

Proposition 1. *Let $(M^5; \varphi, \xi, \eta, g)$ be a 5-dimensional Einstein-Sasaki manifold. Denote by $W: \bigwedge^2(T^h) \rightarrow \bigwedge^2(T^h)$ the Weyl tensor on the horizontal bundle. Then W is anti-selfdual with respect to the algebraic Hodge operator of the bundle T^h , i.e. $*W = -W$.*

Proof. We fix an orthonormal basis $e_1, e_2 = \varphi(e_1), e_3, e_4 = \varphi(e_3)$ in T^h . By the rule $\varphi(X \wedge Y) = \varphi(X) \wedge \varphi(Y)$, φ acts on $\bigwedge^2(T^h) = \bigwedge^2_+(T^h) \oplus \bigwedge^2_-(T^h)$ and we see immediately that in the basis $\{e_1 \wedge e_2 + e_3 \wedge e_4, e_1 \wedge e_3 - e_2 \wedge e_4, e_1 \wedge e_4 + e_2 \wedge e_3\}$ of $\bigwedge^2_+(T^h)$ the matrix representation of φ is given by

$$\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Since the curvature tensor commutes with the transformation φ in a Sasaki manifold, the Weyl tensor $W: \bigwedge^2(T^h) \rightarrow \bigwedge^2(T^h)$ also commutes with φ . Consequently, we obtain for $W_+: \bigwedge^2_+(T^h) \rightarrow \bigwedge^2_+(T^h)$ the matrix representation

$$W_+ = \begin{pmatrix} A & 0 & 0 \\ 0 & B & D \\ 0 & D & C \end{pmatrix}$$

with

$$\begin{aligned} A &= W_{1212} + 2W_{1234} + W_{3434}, & B &= W_{1313} - 2W_{1324} + W_{2424}, \\ C &= W_{1414} + 2W_{1423} + W_{2323}, & D &= -2(W_{2414} + W_{2423}). \end{aligned}$$

We prove $A = B = C = D = 0$. In fact, since M^5 is an Einstein space with scalar curvature $R = 20$, we have

$$W_{1212} = R_{1212} + 1, \quad W_{3434} = R_{3434} + 1, \quad W_{1234} = R_{1234},$$

and taking into account $R_{1551} = 1$ ($e_5 = \xi$) we obtain

$$\begin{aligned} A &= R_{1212} + R_{3434} + 2R_{1234} + 2 \\ &= (-R_{1221} - R_{1331} - R_{1441} - R_{1551}) \\ &\quad + (-R_{4114} - R_{4224} - R_{4334} - R_{4554}) \\ &\quad + R_{1331} + R_{1441} + R_{4114} + R_{4224} + 2R_{1234} + 4 \\ &= -R_{11} - R_{44} + 2(R_{1331} + R_{1441} + R_{1234}) + 4 \\ &= -8 + 2(R_{1331} + R_{1441} + R_{1234}) + 4. \end{aligned}$$

The Muskal-Okumara lemma (see e.g. [3, p. 93]) now yields

$$R_{1234} + R_{1331} + R_{1441} = -d\eta(e_3, e_4)g(e_1, e_1) = -2g(e_3, \varphi(e_4)) = 2$$

and we finally have $A = 0$. In the same way we prove $B = C = 0$. Finally, we calculate D —using once again the Einstein equation and the Muskal-Okumara formula—

$$D = -2(W_{2414} + W_{2423}) = -2(R_{2414} + R_{2423}) = 0.$$

3. The $SU(2)$ -reduction defined by a nonvanishing spinor

Consider the group $\text{Spin}(5)$ and its complex spinor representation $\kappa: \text{Spin}(5) \rightarrow \text{GL}(\Delta_5)$. $\text{Spin}(5)$ acts transitively on the 7-dimensional sphere $S(\Delta_5) = \{\psi \in \Delta_5: |\psi| = 1\}$. The isotropy group $\hat{H}(\psi)$ of a fixed spinor $\psi \neq 0$ is a subgroup $\hat{H}(\psi) \subset \text{Spin}(5)$ which projects one-to-one onto a subgroup $H(\psi) \subset \text{SO}(5)$ which is conjugate to $SU(2) \subset \text{SO}(5)$. We fix an orthonormal basis e_1, \dots, e_5 in R^5 and identify Δ_5 with $\mathbb{C}^2 \otimes \mathbb{C}^2$. Let us introduce the basis $u(\varepsilon_1, \varepsilon_2)$ in Δ_5 (see e.g. [4]):

$$u(\varepsilon_1, \varepsilon_2) = u(\varepsilon_1) \otimes (\varepsilon_2), \quad \text{with } u(1) = \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad u(-1) = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Denote by g_1, g_2 and T the matrices

$$g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad T = g_1 g_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The Clifford multiplication of a vector by a spinor is then defined by

$$\begin{aligned} e_1 &= I \otimes g_1, & e_2 &= I \otimes g_2, & e_3 &= i g_1 \otimes T, \\ e_4 &= i g_2 \otimes T, & e_5 &= -i T \otimes T. \end{aligned}$$

The Lie algebra $\hat{\mathfrak{h}}$ of the isotropy group of the spinor $\psi_0 = u(1, 1)$ is given by

$$\hat{\mathfrak{h}} = \{ \alpha \in \text{spin}(5) : \alpha \cdot u(1, 1) = 0 \}$$

$$= \left\{ \sum_{1 \leq i < j \leq 5} w_{ij} e_i e_j : \begin{array}{ll} w_{12} + w_{34} = 0 & w_{14} + w_{23} = 0 \\ w_{13} - w_{24} = 0 & w_{15} = w_{25} = w_{35} = w_{45} = 0 \end{array} \right\}.$$

Using this concrete realization of the spin-representation one immediately proves

Lemma 1. (a) *Let $\psi_1, \psi_2 \in S(\Delta_5)$ be two orthogonal spinors of length one and suppose that for the corresponding Lie algebras $\hat{\mathfrak{h}}(\psi_1) \cap \hat{\mathfrak{h}}(\psi_2) \neq \{0\}$. Then for each vector $X \in R^5$ it holds that*

$$(\psi_1, X \cdot \psi_2) = 0,$$

where $X \cdot \psi_2$ denotes the Clifford multiplication of the vector X by the spinor ψ_2 .

(b) *For each spinor $\psi \neq 0$ there exists a unique vector $\xi \in R^5$ of length one such that $\xi \cdot \psi = i\psi$.*

Denote by $\pi: Q \rightarrow M^5$ the frame bundle of the oriented Riemannian manifold (M^5, g) and let $\pi: P \rightarrow M^5$ be a spin-structure. If $\psi \in \Gamma(S)$ is a section of length one in the spinor-bundle $S = P \times_{\kappa} \Delta_5$, then we consider

$$P^0 = \{ p \in P : \psi(\pi(p)) = [p, u(1, 1)] \}.$$

Since $\text{Spin}(5)$ acts transitively on $S(\Delta_5)$ with isotropy group $\hat{H}(\psi_0) = \text{SU}(2)$, P^0 is a $\text{SU}(2)$ -principal fiber bundle over M^5 . Denote by $\lambda: P \rightarrow Q$ the two-fold covering of the spin structure over the frame bundle. Then $\lambda|_{P^0}: P^0 \rightarrow \lambda(P^0) = Q^0$ is bijective and, consequently, we obtain an $\text{SU}(2)$ -reduction $Q^0 \subset Q$ of the frame bundle Q . We now investigate the topological type of this reduction in the case that M^5 is simply-connected. The classifying space of the group $\text{SU}(2) = \text{Sp}(1)$ is $P^\infty(H)$, a CW-complex of the type $e^0 \cup e^4 \cup e^8 \cup \dots$. Since M^5 is a 5-dimensional CW-complex we see that the isomophy classes of $\text{SU}(2)$ -bundles over M^5 correspond to the set $[M^5, P^\infty(H)] = [M^5, S^4]$. Using the classification theorem of Steenrod (see e.g. [18]) we obtain

$$[M^5; S^4] = \frac{H^5(M^5; Z_2)}{\text{Sq}^2 \mu_* H^3(M^5; Z)}$$

where $\mu_*: H^3(M^5; Z) \rightarrow H^3(M^5; Z_2)$ is the Z_2 -reduction and Sq^2 denotes the second Steenrod square. Since M^5 is a spin-manifold its second Stiefel-Whitney class vanishes and, consequently, (look, for example, into the Wu-formula!) $\text{Sq}^2 = 0$. Therefore, on a 5-dimensional, compact, simply-connected spin-manifold M^5 there are precisely two $\text{SU}(2)$ -principal fiber bundles:

$$[M^5, S^4] = H^5(M^5; Z_2) = Z_2.$$

Theorem 1. *Let M^5 be a 5-dimensional, compact simply-connected spin-manifold with a nowhere vanishing spinor field $\psi \in \Gamma(S)$. Then the following conditions are equivalent:*

- (1) Q^0 is the trivial $SU(2)$ -principal fiber bundle.
- (2) The subbundle $T^h = Q^0 \times_{SU(2)} R^4 \subset TM^5$ is trivial.
- (3) M^5 is parallelizable.
- (4) $\dim H_2(M^5; Z_2) \equiv 1 \pmod{2}$.

On the other hand Q^0 is a nontrivial $SU(2)$ -principal fiber bundle if and only if $\dim H_2(M^5; Z_2) \equiv 0 \pmod{2}$.

Proof. The implications (1) \Rightarrow (2) \Rightarrow (3) are trivial, (3) \Rightarrow (4) follows from classical results concerning vector fields on spin-manifolds (see [20]). Suppose now that $\dim H_2(M^5; Z_2) \equiv 1 \pmod{2}$ and fix a point $m_0 \in M^5$. The space $M^5 \setminus \{m_0\}$ has the homotopy type of a 4-dimensional CW-complex and $\pi_1(M^5) = 0$ implies $H^4(M^5 \setminus \{m_0\}; Z) = 0$. Using the Hopf Classification Theorem we obtain

$$[M^5 \setminus \{m_0\}; P^\infty(H)] = [M^5 \setminus \{m_0\}; S^4] = H^4(M^5 \setminus \{m_0\}; Z) = 0.$$

This means that the bundle Q^0 is trivial over $M^5 \setminus \{m_0\}$. Consider a section $X^* = (X_1, \dots, X_5)$ in Q^0 over $M^5 \setminus \{m_0\}$. The index $\text{Ind}(X^*)$ is an element of $\pi_4(SU(2)) = Z_2$. Furthermore, if $\text{Ind}(X^*) = 0$ then Q^0 is a trivial bundle over M^5 . We calculate the index of X^* in the following way: Look at the pair (X_1, X_2) of vector fields on $M^5 \setminus \{m_0\}$ and its index $\text{Ind}(X_1, X_2) \in \pi_4(V_{5,2}) = Z_2$. An easy homotopy argument shows that the map $f: SU(2) \rightarrow SO(4) \rightarrow SO(5) \rightarrow V_{5,2} = SO(5)/SO(2)$ induces an isomorphism $f_\#: \pi_4(SU(2)) \rightarrow \pi_4(V_{5,2})$. Consequently, $\text{Ind}(X^*)$ vanishes in $\pi_4(SU(2))$ if and only if $\text{Ind}(X_1, X_2)$ vanishes in $\pi_4(V_{5,2})$. Now the index of a pair of vector fields with isolated singularities is well known (see e.g. [20]):

$$\begin{aligned} \text{Ind}(X_1, X_2) &= \sum_{i=0}^2 \dim H_i(M^5; Z_2) \\ &= 1 + \dim H_2(M^5; Z_2) \pmod{2}. \end{aligned}$$

This proves the implication (4) \Rightarrow (1).

Remark. Using similar techniques one can show that in case the $SU(2)$ -reduction $Q^0 \subset Q$ is nontrivial it does *not* admit a reduction to the subgroup $U(1) \subset SU(2)$.

4. The Einstein-Sasaki structure defined by a Killing spinor

Let $\psi \in \Gamma(S)$ be an eigenspinor of the Dirac operator corresponding to the eigenvalue $\pm \frac{1}{4} \sqrt{5R}$ on a compact, 5-dimensional Einstein spin-manifold M^5

with positive scalar curvature R ,

$$D\psi = \pm \frac{1}{4}\sqrt{5R}\psi.$$

Then ψ satisfies a stronger equation, namely

$$\nabla_X\psi = \mp \frac{\sqrt{R}}{4\sqrt{5}}X \cdot \psi,$$

where $X \cdot \psi$ denotes the Clifford multiplication of the vector X by the spinor ψ (see e.g. [4]). Such spinor fields are sometimes called Killing spinors (see e.g. [9]). It is well known that the length $|\psi|$ of ψ is constant.

Denote by $E_{\pm} \subset L^2(S)$ the eigenspace of the Dirac operator corresponding to the eigenvalues $\pm \frac{1}{4}\sqrt{5R}$, respectively.

Proposition 2. *If M^5 is not conformally flat then $\dim E_{\pm} \leq 1$.*

Proof. Suppose we have two solutions ψ_1, ψ_2 satisfying

$$\nabla_X\psi_i = -\frac{1}{4\sqrt{5}}\sqrt{R}X \cdot \psi_i \quad (i = 1, 2).$$

Without loss of generality we can assume that $(\psi_1, \psi_2) \equiv 0$ since $X(\psi_1, \psi_2) = (\nabla_X\psi_1, \psi_2) + (\psi_1, \nabla_X\psi_2) = 0$.

Fix a point $m_0 \in M^5$ such that the Weyl tensor does not vanish at m_0 . Then we have for any 2-form $\eta^2 \in \Lambda^2$

$$W(\eta^2) \cdot \psi_1 = 0 = W(\eta^2) \cdot \psi_2,$$

where $W: \Lambda^2(TM^5) \rightarrow \Lambda^2(TM^5)$ is the Weyl tensor (see e.g. [5]). Since $W \neq 0$ at m_0 we apply Lemma 1 and conclude $(\psi_1, X \cdot \psi_2) = 0$ for any vector $X \in T_{m_0}(M^5)$. Consider a local frame $s = (s_1, \dots, s_5)$ in the $SU(2)$ -bundle $Q^0 \subset Q$ corresponding to ψ_1 as well as the section s^* in the reduction P^0 of the spin-structure P . Then we have (locally) $\psi_1 = [s^*, u(1, 1)]$ and $(\psi_1, X \cdot \psi_2) = 0$ for each vector X implies $\psi_2 = [s^*, z \cdot u(-1, -1)]$ with a complex valued function z . Consequently, we obtain

$$\begin{aligned} \nabla_X\psi_1 &= \frac{1}{2} \sum_{i < j} w_{ij}(X)e_i e_j u(1, 1) = -\frac{1}{4\sqrt{5}}\sqrt{R}X \cdot u(1, 1), \\ \nabla_X\psi_2 &= dz(X) \cdot u(-1, -1) + \frac{1}{2} \sum_{i < j} w_{ij}(X)e_i e_j u(-1, -1) \\ &= -\frac{1}{4\sqrt{5}}\sqrt{R}X \cdot u(-1, -1), \end{aligned}$$

where w_{ij} are the connection forms of the Riemannian manifold M^5 with respect to the frame s . Using the formulas for the Clifford multiplication

given above we conclude in particular ($X = s_1$)

$$\begin{aligned} -w_{15}(s_1) + iw_{25}(s_1) &= i\frac{1}{2\sqrt{5}}\sqrt{R}, \\ -w_{15}(s_1) - iw_{25}(s_1) &= i\frac{1}{2\sqrt{5}}\sqrt{R}, \end{aligned}$$

thus a contradiction.

Remark. Consider a Killing spinor ψ with

$$\nabla_X\psi = \lambda X \cdot \psi \quad \left(\lambda = \pm \frac{\sqrt{R}}{4\sqrt{5}} \right)$$

and the corresponding $SU(2)$ -reduction Q^0 of the frame bundle Q . If s is a local section in Q^0 we have

$$\frac{1}{2} \sum_{i < j} w_{ij}(X) e_i e_j u(1, 1) = \lambda X \cdot u(1, 1).$$

Denote by $\sigma^1, \dots, \sigma^5$ the dual frame to s_1, \dots, s_5 . Then an algebraic calculation yields the following formulas:

$$\begin{aligned} w_{12} + w_{34} &= 2\lambda\sigma^5, & w_{13} - w_{24} &= 0, & w_{14} + w_{23} &= 0, \\ w_{15} &= -2\lambda\sigma^2, & w_{25} &= 2\lambda\sigma^1, & w_{35} &= -2\lambda\sigma^4, & w_{45} &= 2\lambda\sigma^3. \end{aligned}$$

We consider now an Einstein space (M^5, g) such that $R = 20$ as well as a Killing spinor ψ satisfying $\nabla_X\psi = -\frac{1}{2}X \cdot \psi$. According to Lemma 1 there exists a unique vector field ξ of length one such that $\xi \cdot \psi = i\psi$. Furthermore, we define a 1-form η by $\eta(X) = (X \cdot \psi, \psi)/i$ and a (1,1)-tensor $\varphi := -\nabla\xi$.

Theorem 2. *Let (M^5, g) be an Einstein space with scalar curvature $R = 20$ and Killing spinor ψ . Then $(M^5; \varphi, \xi, \eta, g)$ is an Einstein-Sasaki manifold.*

Proof. We must check the conditions (a)–(g) defining a Sasaki structure in our situation. For the local calculations we choose a frame s in the $SU(2)$ -reduction. We have

$$\begin{aligned} d\eta(X, Y) &= \frac{1}{i} \{X(Y\psi, \psi) - Y(X\psi, \psi) - ([X, Y]\psi, \psi)\} \\ &= \frac{1}{i} \{(Y\nabla_X\psi, \psi) + (Y\psi, \nabla_X\psi) - (X\nabla_Y\psi, \psi) - (X\psi, \nabla_Y\psi)\} \\ &= -\frac{1}{i} ((YX - XY)\psi, \psi) \end{aligned}$$

and, consequently,

$$d\eta = 2(\sigma^1 \wedge \sigma^2 + \sigma^3 \wedge \sigma^4).$$

This implies immediately

$$\eta \wedge d\eta \wedge d\eta = 8dM^5.$$

The equation $\eta(\xi) = 1$ follows directly from the definition of ξ and η . We differentiate the equation $\xi \cdot \psi = i\psi$ and obtain

$$\begin{aligned}(\nabla_X \xi) \cdot \psi + \xi \nabla_X \psi &= i \nabla_X \psi \\ -\varphi(X) \cdot \psi - \frac{1}{2} \xi X \psi &= -\frac{i}{2} X \cdot \psi.\end{aligned}$$

In particular we have $\varphi(X)\psi = iX \cdot \psi$ for each X orthogonal to ξ . Replacing X by $\varphi(X)$ we have

$$-\varphi^2(X) \cdot \psi - \frac{1}{2} \xi \varphi(X)\psi = -\frac{i}{2} \varphi(X) \cdot \psi.$$

Combining the last two equations we obtain

$$-\varphi^2(X) \cdot \psi - \frac{1}{2}(X + i\xi X)\psi = 0.$$

If X is parallel to ξ it follows that $\varphi^2(X) \cdot \psi = 0$ and, consequently, $\varphi^2(X) = 0$. If X is orthogonal to ξ we have $\frac{1}{2}(X + i\xi X)\psi = \frac{1}{2}(X - iX\xi)\psi = \frac{1}{2}(X - i^2 X)\psi = X \cdot \psi$ and

$$\{\varphi^2(X) + X\} \cdot \psi = 0.$$

The last formula implies $\varphi^2(X) = -X$ in case X is orthogonal to ξ . Summing up we proved $\varphi^2 = -\text{Id} + \eta \otimes \xi$.

We prove now that ξ is a Killing vector field, i.e. φ is antisymmetric. We already know

$$\varphi(X) \cdot \psi + \frac{1}{2} \xi X \psi = \frac{i}{2} X \cdot \psi.$$

We multiply by $Y \cdot \psi$ from the right and left side:

$$\begin{aligned}(\varphi(X)\psi, Y \cdot \psi) + \frac{1}{2}(\xi X \psi, Y \psi) &= \frac{i}{2}(X\psi, Y\psi), \\ (Y\psi, \varphi(X)\psi) + \frac{1}{2}(Y\psi, \xi X \psi) &= -\frac{i}{2}(Y\psi, X\psi).\end{aligned}$$

Taking into account $Y \cdot \varphi(X) + \varphi(X) \cdot Y = -2g(Y, \varphi(X))$ we obtain

$$2g(Y, \varphi(X))|\psi|^2 + \text{Re}((\xi X \psi, Y \psi)) = -\text{Im}(X\psi, Y\psi).$$

Finally we remark that the real part of $(\xi X \psi, Y \psi)$ and the imaginary part of $(X\psi, Y\psi)$ are antisymmetric in X and Y . It follows that

$$g(Y, \varphi(X)) = -g(X, \varphi(Y)),$$

i.e. ξ is a Killing vector field.

The equation $g(\varphi(X), \varphi(Y)) = g(X, Y) - \eta(X)\eta(Y)$ is now a formal consequence of some formulas we already proved:

$$\begin{aligned} g(\varphi(X), \varphi(Y)) &= -g(\varphi^2(X), Y) \\ &= -g(-X + \eta(X)\xi, Y) = g(X, Y) - \eta(X)g(\xi, Y) \\ &= g(X, Y) - \eta(X)\eta(Y). \end{aligned}$$

We prove the property $d\eta(X, Y) = 2g(X, \varphi(Y))$ —using the fact that ξ is a Killing field—as follows:

$$\begin{aligned} d\eta(X, Y) &= X\eta(Y) - Y\eta(X) - \eta[X, Y] \\ &= Xg(\xi, Y) - Yg(\xi, X) - g(\xi, [X, Y]) = g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X) \\ &= -g(\varphi(X), Y) + g(X, \varphi(Y)) = 2g(X, \varphi(Y)). \end{aligned}$$

It remains to prove the integrability condition $(\nabla_Y \varphi)(X) = g(X, Y)\xi - \eta(X)Y$. We again start with $\varphi(X) \cdot \psi = \frac{1}{2}(iX - \xi X) \cdot \psi$ and differentiate this equation:

$$\begin{aligned} \nabla_Y(\varphi(X)) \cdot \psi - \frac{1}{2}\varphi(X)Y\psi &= \frac{1}{2}(i\nabla_Y X - \nabla_Y \xi \cdot X - \xi \nabla_Y X) \cdot \psi \\ &\quad + \frac{1}{2}(iX - \xi X)(-\frac{1}{2}Y\psi). \end{aligned}$$

On the other hand we have

$$\varphi(\nabla_Y X)\psi = \frac{1}{2}(i\nabla_Y X - \xi \nabla_Y X)\psi.$$

This implies

$$(\nabla_Y \varphi)(X) \cdot \psi = \frac{1}{2} \left\{ \varphi(X)Y + \varphi(Y)X + \frac{\xi XY - iXY}{2} \right\} \psi.$$

First of all we consider the case that X and Y are orthogonal to ξ . Then $(\xi XY - iXY)\psi = 0$ and $\varphi(X) \cdot \psi = \frac{1}{2}(iX - \xi X)\psi = iX\psi$. In this situation we have

$$\begin{aligned} (\nabla_Y \varphi)(X)\psi &= \frac{1}{2}\{\varphi(X)Y + \varphi(Y)X\}\psi \\ &= \frac{1}{2}\{-Y\varphi(X) - 2g(Y, \varphi(X)) - X\varphi(Y) - 2g(X, \varphi(Y))\}\psi \\ &= \frac{1}{2}\{-iYX - iXY\}\psi = g(X, Y)\xi \cdot \psi \end{aligned}$$

and finally $(\nabla_Y \varphi)(X) = g(X, Y)\xi$.

The second case we want to consider is $X = \xi$. Then

$$\begin{aligned} (\nabla_Y \varphi)(X) &= \nabla_Y(\varphi(\xi)) - \varphi(\nabla_Y \xi) = \varphi^2(Y) = -Y + \eta(Y)\xi \\ &= g(\xi, Y)\xi - g(\xi, X)Y = g(X, Y)\xi - \eta(X)Y. \end{aligned}$$

If $Y = \xi$ we have

$$\{\xi XY - iXY\}\psi = \{-X\xi\xi - iX\xi\}\psi = \{X + X\}\psi = 2X \cdot \psi,$$

(X orthogonal to ξ) and it follows that

$$(\nabla_Y \varphi)(X) \cdot \psi = \frac{1}{2} \{ \varphi(X)\xi + X\} \psi = \frac{1}{2} \{ i\varphi(X) + X\} \psi = \frac{1}{2} \{ i^2 X + X\} \psi = 0.$$

The last equation implies

$$(\nabla_\xi \varphi)(X) = 0 = g(\xi, X)\xi - \eta(X)\xi$$

for each X orthogonal to ξ . Last but not least we consider the case $X = Y = \xi$. Then we have

$$\begin{aligned} (\nabla_\xi \varphi)(\xi) &= \nabla_\xi(\varphi(\xi)) - \varphi(\nabla_\xi \xi) \\ &= 0 - \varphi^2(\xi) = 0 = g(\xi, \xi)\xi - \eta(\xi)\xi \end{aligned}$$

and the integrability condition is proved.

Remark 1. The existence of a Killing spinor ψ imposes algebraic conditions on the Weyl tensor W , namely $W(\eta^2) \cdot \psi = 0$ for any 2-forms η^2 . In the case of dimension five this implies

$$\sum_{1 \leq i < j \leq 5} W_{ij} e_i e_j u(1, 1) = 0.$$

Taking into account the structure of the Lie algebra \hat{h} described in §3 we conclude

$$W_{12} + W_{34} = 0, \quad W_{13} - W_{24} = 0, \quad W_{14} + W_{23} = 0, \quad W_{i5} = 0,$$

and this is precisely the anti-selfduality condition for the Weyl tensor

$$W: \bigwedge^2(T^h) \rightarrow \bigwedge^2(T^h),$$

which is satisfied automatically in any Einstein-Sasaki space (Proposition 1).

Remark 2. Using the properties of the Sasaki structure we have in particular for the Lie-derivative:

$$\mathcal{L}_\xi \eta = 0, \quad \mathcal{L}_\xi(d\eta) = 0, \quad \mathcal{L}_\xi \varphi = 0.$$

Remark 3. Obviously, if we start with a spinor satisfying $\nabla_X \psi = \frac{1}{2} X \cdot \psi$ we obtain in the same way an Einstein-Sasaki structure.

5. A simply-connected Einstein-Sasaki manifold admits a Killing spinor

Theorem 3. *Let $(M^5; \varphi, \xi, \eta, g)$ be a simply-connected Einstein-Sasaki manifold, with spin-structure. Then the equations $\nabla_X \psi = \pm \frac{1}{2} X \cdot \psi$ have nontrivial solutions.*

Proof. Consider the subbundle E of the spinor bundle S defined by

$$E = \{\psi \in S: \xi\psi = i\psi, \{2\varphi(X) + \xi X - iX\}\psi = 0 \\ \text{for each vector } X \in TM^5\}.$$

Using the algebraic description of Δ_5 given above it is easy to see that E is a 1-dimensional complex subbundle of S . We introduce a covariant derivative $\tilde{\nabla}: \Gamma(E) \rightarrow \Gamma(T^* \otimes E)$ in E by the formula

$$\tilde{\nabla}_X \psi = \nabla_X \psi + \frac{1}{2} X \cdot \psi.$$

First of all we must prove that $\tilde{\nabla}_X \psi$ is a section in E if ψ belongs to $\Gamma(E)$.

Suppose that $\xi\psi = i\psi$ and $\{2\varphi(X) + \xi X - iX\}\psi = 0$. Then

$$\begin{aligned} \nabla_Y \xi \cdot \psi + \xi \nabla_Y \psi &= i \nabla_Y \psi, \\ \nabla_Y \xi \psi + \xi (\nabla_Y \psi + \frac{1}{2} Y \psi) - \frac{1}{2} \xi Y \psi &= i (\nabla_Y \psi + \frac{1}{2} Y \psi) - \frac{1}{2} i Y \psi, \\ \frac{1}{2} (2 \nabla_Y \xi - \xi Y + i Y) \psi + \xi (\tilde{\nabla}_Y \psi) &= i (\tilde{\nabla}_Y \psi). \end{aligned}$$

Since we have a Sasaki structure it holds that $\nabla_Y \xi = -\varphi(Y)$. ψ is a section in E . This implies

$$\xi(\tilde{\nabla}_Y \psi) = i(\tilde{\nabla}_Y \psi).$$

In the same way we prove the second algebraic condition for $\tilde{\nabla}_Y \psi$. We differentiate the equation

$$\{2\varphi(X) + \xi X - iX\}\psi = 0$$

with respect to Y and we use the Sasaki conditions $\varphi = -\nabla \xi$, $(\nabla_Y \varphi)(X) = g(X, Y)\xi - \eta(X)Y$. After some obvious calculations we obtain

$$\begin{aligned} \left\{ 2g(X, Y)\xi - 2\eta(X)Y - \varphi(Y)X - \varphi(X)Y - \frac{\xi XY - iXY}{2} \right\} \psi \\ + \{2\varphi(X) + \xi X - iX\} \tilde{\nabla}_Y \psi = 0. \end{aligned}$$

The first term vanishes. Consider for example the case that X and Y are orthogonal to ξ . Then we have $\{\xi XY - iXY\}\psi = 0$ with respect to $\xi\psi = i\psi$ and, consequently, the first term reduces to

$$\begin{aligned} \{2g(X, Y)\xi - \varphi(Y)X - \varphi(X)Y\} \psi \\ = \{2g(X, Y)i + (2g(\varphi(Y), X) + X\varphi(Y)) + (2g(\varphi(X), Y) + Y\varphi(X))\} \psi \\ = \{2g(X, Y)i + X\varphi(Y) + Y\varphi(X)\} \psi. \end{aligned}$$

Since ψ is a section in E , we have

$$\{2\varphi(X) + \xi X - iX\}\psi = 0.$$

If X is orthogonal to ξ we obtain

$$\varphi(X) \cdot \psi = iX \cdot \psi.$$

The first term mentioned above thus eventually reduces to

$$\{2g(X, Y)i + iXY + iYX\}\psi = 2i\{g(X, Y) - g(X, Y)\}\psi = 0.$$

We handle the cases where X or Y is parallel to ξ in the same way. Then we obtain

$$\{2\varphi(X) + \xi X - iX\}\tilde{\nabla}_Y\psi = 0,$$

i.e. $\tilde{\nabla}_Y\psi$ is a section in E .

The calculation of the curvature tensor \tilde{R} of the connection $\tilde{\nabla}$ in the bundle E yields the formula

$$\begin{aligned} \tilde{R}(X, Y)\psi &= (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X, Y]})\psi + \frac{1}{4}(XY - YX)\psi \\ &= \frac{1}{4} \left(\sum_{i, j} R_{XYij}e_i e_j + XY - YX \right) \psi = \frac{1}{4} \sum_{i, j} W_{XYij}e_i e_j \cdot \psi \end{aligned}$$

with the Weyl tensor W . Here we use the formula

$$W_{ijke} = R_{ijke} + (\delta_{ik}\delta_{je} - \delta_{ie}\delta_{jk})$$

valid in a 5-dimensional Einstein space with scalar curvature $R = 20$. Since M^5 is an Einstein-Sasaki manifold, we have $W(\xi, X) = 0$ and we obtain

$$\tilde{R}(X, Y)\psi = \frac{1}{4} \sum_{i=1}^4 e_i \cdot W(X, Y)e_i \cdot \psi,$$

where $\{e_1, e_2, e_3, e_4\}$ is a frame in T^h orthogonal to ξ . A simple algebraic calculation—using Proposition 1, i.e. $*W = -W$ in $\wedge^2(T^h)$ —now shows

$$\tilde{R}(X, Y)\psi = 0, \quad \psi \in \Gamma(E).$$

Consequently, $(E, \tilde{\nabla})$ is a flat 1-dimensional bundle over a simply-connected manifold M^5 . Thus there exists a $\tilde{\nabla}$ -parallel section ψ in E , i.e. a spinor field satisfying the equation $\nabla_X\psi = -\frac{1}{2}X \cdot \psi$.

Remark. The same procedure allows us to construct a solution of the equation $\nabla_X\psi = +\frac{1}{2}X \cdot \psi$.

Corollary. In case M^5 is simply-connected we have $\dim E_+ = \dim E_-$, where $E_{\pm} \subset L^2(S)$ is the eigenspace of the Dirac operator corresponding to the eigenvalue $\pm\frac{1}{4}\sqrt{5R}$.

6. The classification of compact Einstein spin-manifolds admitting a Killing spinor with regular contact structure

A Sasaki manifold $(M^5; \varphi, \xi, \eta, g)$ is called regular if all integral curves of ξ are closed and have the same length L (see e.g. [3]). In this situation we have an S^1 -action on M^5 and the orbit space is a 4-dimensional manifold X^4 . The projection $\pi: M^5 \rightarrow X^4$ is a principal S^1 -bundle and $2\pi i\eta/L: TM^5 \rightarrow R \cdot i = \mathfrak{S}^1$ is a connection in this bundle. Since $\mathcal{L}_\xi g = 0$ and $\mathcal{L}_\xi \varphi = 0$, X^4 admits a Riemannian metric and an almost complex structure which is integrable (see e.g. [3]). Denote by Ω the Kähler form of X^4 . Then

$$\pi^* \Omega(X, Y) = g(X, \varphi(Y)) = \frac{1}{2} d\eta(X, Y)$$

and we conclude $d\Omega = 0$, i.e. X^4 is a Kähler manifold. Suppose now in addition that M^5 is an Einstein-Sasaki space. The O'Neill formulas yield that X^4 is an Einstein-Kähler manifold with scalar curvature $\mathfrak{R} = \frac{6}{5}R = 24$. Consequently, X^4 is analytically isomorphic to $S^2 \times S^2$, $P^2(\mathbb{C})$ or to one of the del Pezzo surfaces P_k ($3 \leq k \leq 8$; P_k is the surface obtained by blowing up k points in general position in $P^2(\mathbb{C})$, see e.g. [2]). Next we study the topological type of the S^1 -fiber bundle $\pi: M^5 \rightarrow X^4$. The curvature form of the connection $2\pi i\eta/L$ is $\Omega^* = (2\pi i/L)d\eta$. Consequently, the Chern class $c_1^* \in H^2(X^4; \mathbb{R})$ is given by $c_1^* = \Omega^*/2\pi i = d\eta/L$. On the other hand, since X^4 is an Einstein-Kähler manifold its Chern class is given by the Ricci form

$$c_1 = \Omega_{\text{Ric}} = \frac{1}{2\pi} \frac{R}{4} \Omega = \frac{3}{\pi} \Omega = \frac{3}{2\pi} d\eta = \frac{3L}{2\pi} c_1^*$$

and we obtain the relation

$$c_1 = \frac{3L}{2\pi} c_1^*$$

between the Chern class c_1 of X^4 and the Chern class c_1^* of the S^1 -bundle $\pi: M^5 \rightarrow X^4$. X^4 is simply connected. We now apply the Thom-Gysin sequence of the fibration $\pi: M^5 \rightarrow X^4$ and conclude:

- (a) $H^1(M^5; \mathbb{Z}) = 0$ (since $c_1^* \neq 0$).
- (b) $H^4(M^5; \mathbb{Z}) = H^4(X^4; \mathbb{Z})/c_1^* \cup H^2(X^4; \mathbb{Z})$.
- (c) $0 = w_2(M^5) = \pi^* w_2(X^4)$. If $w_2(X^4) \neq 0$ then $c_1^* \equiv w_2(X^4) \equiv c_1 \pmod{2}$.

In case $w_2(X^4) \neq 0$ the spin structure of M^5 implies an additional condition, namely

$$\frac{1}{2} \left(1 - \frac{2\pi}{3L} \right) c_1(X^4) \in H^2(X; \mathbb{Z}).$$

- (d) The Killing spinor ψ on M^5 defines an $SU(2)$ -reduction Q^0 of the frame bundle. Consequently, we have an isomorphism

$$\pi^* T_{\mathbb{C}} X^4 = T^h = Q^0 \times_{SU(2)} \mathbb{C}^2$$

of 2-dimensional complex vector bundles. This isomorphism yields $\pi^*c_1(X^4) = 0$ because the first Chern class of any $SU(2)$ -bundle vanishes. The Thom-Gysin sequence imposes a further restriction: $c_1/c_1^* \in Z$.

We now classify all possible Einstein spaces M^5 .

First case: $X^4 = P^2(\mathbb{C})$. If X^4 is analytically isomorphic to $P^2(\mathbb{C})$ and admits an Einstein-Kähler metric then X^4 is analytically isometric to $P^2(\mathbb{C})$ (see e.g. [15]). The cohomology algebra $H^*(P^2(\mathbb{C}))$ is isomorphic to $Z[\alpha]/(\alpha^3)$ and the first Chern class is given by the $c_1 = 3\alpha$, $\alpha \in H^2(P^2(\mathbb{C}))$. Using the restrictions (c) and (d) stated above we have two possibilities for the Chern class $c_1^* = \alpha, 3\alpha$ with $\pi_1(M^5) = H^4(M^5) = 0$, Z_3 and $L = 2\pi, 2\pi/3$. Since we know the curvature tensor of $P^2(\mathbb{C})$ as well as the curvature form $\Omega^* = (2\pi i/L)d\eta = 4\pi i\Omega/L$ of the Riemannian submersion $\pi: M^5 \rightarrow X^4$ we can apply the O'Neill formulas again and conclude that M^5 is conformally flat. Consequently, M^5 is isometric to S^5 in case $c_1^* = \alpha$ and isometric to S^5/Z_3 in case $c_1^* = 3\alpha$. $P^2(\mathbb{C})$ is a homogeneous Einstein-Kähler manifold. A simple geometric argument shows that we can lift the isometries of $P^2(\mathbb{C})$ to isometries of M^5 , i.e. $M^5 = S^5/Z_3$ is the homogeneous space of constant curvature one and fundamental group $\pi_1(M^5) = Z_3$.

Second case: $X^4 = S^2 \times S^2$. Suppose that X^4 is analytically isomorphic to $S^2 \times S^2 = \hat{G}_{4,2} = Q_2 =$ the Klein quadric in $P^3(\mathbb{C})$. Moreover, X^4 has an Einstein-Kähler metric with positive scalar curvature. Then the Lie algebra \mathfrak{h} of all holomorphic vector fields on X^4 is the complexification of the Lie algebra \mathfrak{i} of all Killing vector fields (see [14]) and we conclude that $\dim_{\mathbb{R}} \mathfrak{i} = \dim_{\mathbb{C}} \mathfrak{h} = 6$, i.e. X^4 admits a 6-dimensional group of isometries. We now apply a result of L. Berard Bergery (see e.g. [1]) stating in our situation that X^4 is a symmetric Einstein-Kähler structure on $S^2 \times S^2$. Consequently, X^4 is analytically isometric to $S^2 \times S^2$. The cohomology algebra of $S^2 \times S^2$ is $H^*(S^2 \times S^2) = \Lambda(\alpha, \beta)$ and its first Chern class is given by $c_1 = 2(\alpha + \beta)$. We again have two possibilities $c_1^* = (\alpha + \beta), 2(\alpha + \beta)$ with $\pi_1(M^5) = H^4(M^5) = 0$, Z_2 and $L = 4\pi/3, 2\pi/3$.

Now we study the geometry of the Riemannian submersion $\pi: M^5 \rightarrow X^4$ and conclude that M^5 is isometric to the Stiefel manifold $V_{4,2}$ or to $V_{4,2} | Z_2$ with the Einstein metric considered in [11]. The calculation in [4] shows that this space admits a nontrivial Killing spinor.

Third case: $X^4 = P_k$. If X^4 is analytically isomorphic to a del Pezzo surface P_k ($3 \leq k \leq 8$) there is only one possibility for M^5 , namely the simply-connected S^1 -fiber bundle over P_k . Indeed, the cohomology algebra of P_k is generated by elements $\alpha, E_1, \dots, E_k \in H^2(P_k)$ and the first Chern class is given by

$$c_1(P_k) = 3\alpha + E_1 + \dots + E_k$$

(see e.g. [2]). Using the restriction for c_1^* given above we see that there remains only one possibility,

$$c_1^* = 3\alpha + E_1 + \cdots + E_k$$

with $\pi_1(M^5) = H^4(M^5) = H^4(P_k)/c_1^* \cup H^2(P_k) = 0$.

Summing up we proved the following

Theorem 4. *Let (M^5, g) be an Einstein space with Killing spinor ψ and scalar curvature $R = 20$. Suppose in addition that the associated contact structure is regular. Then there are three possibilities:*

(1) M^5 is isometric to S^5 or S^5/Z_3 with the homogeneous metric of constant curvature.

(2) M^5 is isometric to the Stiefel manifold $V_{4,2}$ or $V_{4,2}/Z_2$ with the Einstein metric considered in [11],[4].

(3) M^5 is diffeomorphic to the simply-connected S^1 -fiber bundle with Chern class $c_1^* = c_1(P_k)$ over a del Pezzo surface P_k ($3 \leq k \leq 8$).

Remark. S. Sulanke (see [19]) classified all spaces S^5/Γ of constant curvature with a Killing spinor. It turned out that except for the case S^5/Z_3 all other examples defined a nonregular contact structure. The integral curves of ξ are all closed but have different length. It seems to be interesting, using higher-dimensional Seifert-fibrations, to classify all Einstein spaces with Killing spinors such that the integral curves are closed, but with different length. The orbit space X^4 in this case is smooth except for a finite number of points.

References

- [1] L. Berard Bergery, *Sur de nouvelles variétés riemanniennes d'Einstein* (preprint, no. 839), Inst. Élie Cartan, 1981.
- [2] A. Besse, *Geometric riemannienne en dimension 4*, Cedec, Paris, 1981.
- [3] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Notes in Math., Vol. 509, Springer, Berlin, 1976.
- [4] Th. Friedrich, *Der erste Eigenwert des Dirac-Operators einer kompakten Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung*, Math. Nachr. **97** (1980) 117-146.
- [5] —, *A remark on the first eigenvalue of the Dirac operator on 4-dimensional manifolds*, Math. Nachr. **102** (1981) 53-56.
- [6] Th. Friedrich & R. Grunewald, *On Einstein metrics on the twistor space of a four-dimensional Riemannian manifold*, Math. Nachr. **123** (1985) 55-60.
- [7] —, *On the first eigenvalue of the Dirac operator on 6-dimensional manifolds*, Ann. Global Anal. Geom. **3** (1985) 265-273.
- [8] R. Grunewald, *Untersuchungen über Einstein-Metriken auf 6-dimensionalen Mannigfaltigkeiten*, Diss. A, Humboldt Universität, Berlin, 1987.
- [9] O. Hijazi, *A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors*, Comm. Math. Phys. **104** (1986) 151-162.
- [10] —, *Characterisation de la sphere par les premieres valeurs propres de l'operateur de Dirac en dimension 3,4,7 et 8*, C.R. Acad. Sci. Paris Sér. I Math. **303** (1986) 417-419.

- [11] G. R. Jensen, *Imbeddings of Stiefel manifolds into Grassmannians*, Duke Math. J. **42** (1975) 397–407.
- [12] K. D. Kirchberg, *An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds with positive scalar curvature*, Ann. Global Anal. Geom. **4** (1986) 291–326.
- [13] ———, *Compact six-dimensional Kähler spin manifolds of positive scalar curvature with the smallest possible first eigenvalue of the Dirac operator*, Math. Ann., to appear.
- [14] S. Kobayashi, *Transformation groups in differential geometry*, Springer, Berlin, 1972.
- [15] A. Lascoux & M. Berger, *Variétés Kähleriennes Compactes*, Lecture Notes in Math., Vol. 154, Springer, Berlin, 1970.
- [16] A. Lichnerowicz, *Variétés spinorielles et universalité de l'inégalité de Hijazi*, C. R. Acad. Sci. Paris Sér. I Math. **304** (1987) 227–231.
- [17] B. E. Nilsson & C. N. Pope, *Scalar and Dirac eigenfunctions on the squashed seven-sphere*, Phys. Lett. B **133** (1983) 67–71.
- [18] E. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.
- [19] S. Sulanke, *Der Erste Eigenwert des Dirac-Operators auf S^5/Γ* , Math. Nachr. **99** (1980) 259–271.
- [20] E. Thomas, *The index of a tangent 2-field*, Comment. Math. Helv. **42** (1967) 86–110.
- [21] G. Tian, *On Kähler-Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$* , Invent. Math. **89** (1987) 225–246.
- [22] G. Tian & S. T. Yau, *Kähler-Einstein metrics on complex surfaces with $C_1 > 0$* , Comm. Math. Phys. **112** (1987) 175–203.

HUMBOLDT-UNIVERSITÄT ZU BERLIN